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On the Feasibility of a Generalized  
Linear Program

by  
Hui Hu

TECHNICAL REPORT SOL 89-1

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# On the Feasibility of a Generalized Linear Program

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## Abstract

The first algorithm for solving generalized linear programs was given by ~~George B.~~ Dantzig. His algorithm assumes that a basic feasible solution of the generalized linear program to be solved exists and is given. If the initial basic feasible solution is non-degenerate, then his algorithm is guaranteed to converge. The purpose of this paper is to show how to find an initial basic feasible (possibly degenerate) solution of a generalized linear program by applying the same algorithm to a "phase-one" problem without requiring that the initial basic feasible solution to the latter be non-degenerate. (K<sub>P</sub>) ←

Key Words: linear programming, generalized linear programming, duality.

Abbreviated Title: The feasibility of a GLP

Generalized linear programming is a natural extension of linear programming. The cost coefficients and the columns of the constraint matrix of a generalized linear program may be freely chosen from a compact convex set. Specifically, a generalized linear program is defined as

(GLP) minimize  $cy$   
subject to

$$\begin{aligned} py &= q \\ (c, p^T)^T &\text{ may be freely chosen from } T^1 \times T^m \\ y &\geq 0 \end{aligned}$$

where  $y \geq 0$  is a real number,  $q$  is in  $R^m$ ,  $T^1$  is a closed interval in  $R^1$ , and  $T^m$  is a compact convex set in  $R^m$ .

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Dantzig's algorithm for solving generalized linear programs starts with a basic feasible solution of the problem to be solved, i.e., assumes that one has at hand  $m$  independent vectors  $p^1, p^2, \dots, p^m$  such that  $p^i \in T^m$  and values of scalars  $y_i \geq 0$  such that  $\sum_{i=1}^m p^i y_i = q$  is non-negative. At iteration  $k$ , a new vector  $(c^k, p^{kT})^T \in T^1 \times T^m$  is generated with the most negative reduced cost  $c^k - \pi p^k \leq c - \pi p$  for all  $(c, p^T)^T \in T^1 \times T^m$ , where  $\pi p^{j_i} = c^{j_i}$  for basic  $p^{j_i}$ . Then  $p^k$  is added to the constraint matrix of the  $k$ -th restricted master program with cost coefficient  $c^k$ . If the initial basic feasible solution is non-degenerate, then the optimal solutions of all restricted master programs form a sequence of feasible solutions of (GLP) on which the objective function tends to the value of the program (GLP) (see, e.g., Dantzig (1963), chapters 22 and 24).

The purpose of this paper is to show how to find an initial basic feasible (possibly degenerate) solution of (GLP) by Dantzig's algorithm. Here the idea is same as phase-one of the simplex method. We introduce artificial variables and minimize the summation of all artificial variables. That is, we solve the phase-one problem:

$$(LP) \quad \text{minimize } \sum_{i=1}^m x_i \\ \text{subject to}$$

$$py + \sum_{i=1}^m e^i x_i = q \\ p \text{ may be freely chosen from } T^m \\ y \geq 0, x_i \geq 0 \text{ for } i = 1, \dots, m,$$

where  $e^i$  is the  $i$ -th unit vector in  $R^m$ . We assume, without loss of generality, that  $q_i \geq 0$  for  $i = 1, \dots, m$ , and  $q \neq 0$ . In addition, we assume that the origin is not contained in  $T^m$ . Let  $v(LP)$  denote the optimal value of the objective function of program (LP). Superscripts on vectors denote different vectors, while subscripts on vectors denote different components. To avoid confusion, we state the steps for finding an initial basic feasible solution of (GLP) explicitly.

Initialize.

Let  $k := 0$ ;

let the first restricted master linear program,  $LP(0)$ , be

$$\text{minimize } \sum_{i=1}^m x_i$$

subject to

$$\begin{aligned} \sum_{i=1}^m e^i x_i &= q \\ x_i &\geq 0 \text{ for } i = 1, \dots, m. \end{aligned}$$

(Note that  $v(LP(0)) = \sum_{i=1}^m q_i > 0$  since  $q \geq 0$  and  $q \neq 0$ .)

Step 1.

If  $v(LP(k)) = 0$ , then a basic feasible solution of (GLP) is found, stop.

Else, go to Step 2.

Step 2.

Let  $\pi^k$  be an optimal dual solution of  $LP(k)$ ;

find a  $p^{k+1}$  such that  $\pi^k p^{k+1} = \max\{\pi^k p : p \in T^m\}$ ;

if  $\pi^k p^{k+1} \leq 0$ , then (GLP) is infeasible, stop.

Else, go to Step 3.

Step 3.

Let  $LP(k+1)$  be the linear program

$$\text{minimize } \sum_{i=1}^m x_i$$

subject to

$$\begin{aligned} \sum_{i=1}^{k+1} p^i y_i + \sum_{i=1}^m e^i x_i &= q \\ y_i &\geq 0, i = 1, \dots, k+1, \quad x_i \geq 0, i = 1, \dots, m; \end{aligned}$$

solve  $LP(k+1)$ ;

$k := k + 1$ ;

go to Step 1.

**Theorem.** Suppose that the origin is not contained in  $T^m$ . Then (GLP) is feasible if and only if  $v(LP(k)) = 0$  for some  $k$  or  $v(LP(k)) > 0$  for all  $k$  and  $\lim_{k \rightarrow \infty} v(LP(k)) = 0$ .

**Proof.** Suppose first that  $v(\text{LP}(k)) > 0$  for all  $k$  and  $\lim_{k \rightarrow \infty} v(\text{LP}(k)) = 0$ . Let  $(y_1^k, \dots, y_m^k, x_1^k, \dots, x_m^k)$  be an optimal solution of  $\text{LP}(k)$ . Let  $\Delta_k$  be the index set ordered from low indices  $i$  to high indices such that  $y_i^k > 0$  if and only if  $i \in \Delta_k$ . If we solve  $\text{LP}(k)$  by the simplex method, then  $\Delta_k$  contains no more than  $m$  elements. Let  $\Delta_k = \{i_{k_1}, \dots, i_{k_{|\Delta_k|}}\}$  where  $|\Delta_k|$  denotes the cardinal number of  $\Delta_k$ , and let  $Y^k = (y_{i_{k_1}}^k, \dots, y_{i_{k_{|\Delta_k|}}}^k, 0, \dots, 0) \in R^m$ , that is, the first  $|\Delta_k|$  components of  $Y^k$  are  $y_{i_{k_1}}^k, \dots, y_{i_{k_{|\Delta_k|}}}^k$  and the rest are zeros in the case  $|\Delta_k| < m$ . We claim that  $\{Y^k \in R^m : k = 1, 2, \dots\}$  has a cluster point. Indeed let  $\lambda^k = \sum_{i=1}^m Y_i^k$  for all  $k = 1, 2, \dots$ . Since  $\lambda^k = 0$  implies  $v(\text{LP}(k)) = \sum_{i=1}^m q_i > 0$ , we know that  $\lambda^k = 0$  can only happen for a finite number of  $k$  because an infinite number of times would imply  $\lim_{j \rightarrow \infty} v(\text{LP}(k_j)) = \sum_{i=1}^m q_i > 0$  for a subsequence, contrary to the hypothesis  $\lim_{k \rightarrow \infty} v(\text{LP}(k)) = 0$ . Therefore,  $\lambda^k > 0$  for all  $k > k_0$ . Now let  $s^k \equiv \sum_{i=1}^k p^i y_i^k / \lambda^k \in T^m$  and  $u^k \equiv \sum_{i=1}^m e^i x_i^k$  for all  $k > k_0$ . As  $\lim_{k \rightarrow \infty} v(\text{LP}(k)) = 0$ , we have  $\lim_{k \rightarrow \infty} u^k = 0$  and  $\lim_{k \rightarrow \infty} \lambda^k s^k = q$ . Since  $T^m$  is compact, there exists a subsequence  $s^{k_j}$  such that  $\lim_{j \rightarrow \infty} s^{k_j} = s^*$ . We know that  $s^* \neq 0$  since, by hypothesis, the compact set  $T^m$  does not contain the origin. It follows that  $\lim_{j \rightarrow \infty} \lambda^{k_j} = \|q\| \cdot \|s^*\|^{-1}$  and thus  $\lambda^{k_j}$  is bounded. Therefore, the sequence  $Y^k$  has a cluster point. Let  $A^k = (p^{i_{k_1}}, \dots, p^{i_{k_{|\Delta_k|}}}, 0, \dots, 0) \in R^{m \times m}$  for all  $k > k_0$ . As  $T^m$  is compact, for a subsequence  $k_j$  we have

$$\lim_{j \rightarrow \infty} A^{k_j} = A^*, \quad \lim_{j \rightarrow \infty} Y^{k_j} = Y^*, \quad \text{and} \quad \lim_{j \rightarrow \infty} (A^{k_j} Y^{k_j} + u^{k_j}) = A^* Y^* = q.$$

Now let  $p^{1*}, \dots, p^{r*}$  be the nonzero columns of  $A^*$ ,  $y_{1*}, \dots, y_{m*}$  be the components of  $Y^*$ , and choose  $p^{(r+1)*}, \dots, p^{t*} \in T^m$  such that  $\text{rank}(p^{1*}, \dots, p^{t*}) = m$ . Then  $y_i = y_{i*}$  for  $i = 1, \dots, r$ , and  $y_i = 0$  for  $i = r+1, \dots, t$  is a feasible solution of the system:

$$\begin{aligned} p^{1*} y_1 + \cdots + P^{t*} y_t &= q \\ y_i &\geq 0, \quad i = 1, \dots, t, \end{aligned} \tag{1}$$

and thus we can find a basic feasible solution of (1) by the simplex method. Of course, this basic feasible solution is also a basic feasible solution for (GLP).

It remains to show that  $\lim_{k \rightarrow \infty} v(\text{LP}(k)) > 0$  implies (GLP) infeasible. Since  $\pi^k$  is an optimal dual solution of  $\text{LP}(k)$ ,

$$\pi_i^k \leq 1 \quad \text{for } i = 1, \dots, m, \tag{2}$$

and

$$\pi^k q = \sum_{i=1}^m x_i^k \geq 0 \quad \text{for all } k = 1, 2, \dots \tag{3}$$

Since  $q \geq 0$ , (2) and (3) tell us that the sequence  $\pi^k$  is bounded. The boundedness of  $\pi^k$  implies  $\lim_{k \rightarrow \infty} v(\text{LP}(k)) = v(\text{LP})$  (see, e.g., Dantzig (1963), chapter 24). It follows that  $v(\text{LP}) > 0$  and (GLP) is infeasible. ■

**Remarks.** Given any (GLP) with  $0 \neq q \geq 0$ , one can always generate a sequence of linear programs  $\text{LP}(k)$ . Since  $\{\pi^k : k = 1, 2, \dots\}$  is bounded,  $\lim_{k \rightarrow \infty} v(\text{LP}(k)) = v(\text{LP})$ . Therefore,  $\lim_{k \rightarrow \infty} v(\text{LP}(k)) > 0$  always implies that (GLP) is infeasible. While in the case  $\lim_{k \rightarrow \infty} v(\text{LP}(k)) = 0$ , if  $\limsup_{k \rightarrow \infty} \|s^k\| > 0$ , then  $v(\text{LP}) = 0$  can be attained, which implies that (GLP) is feasible.

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